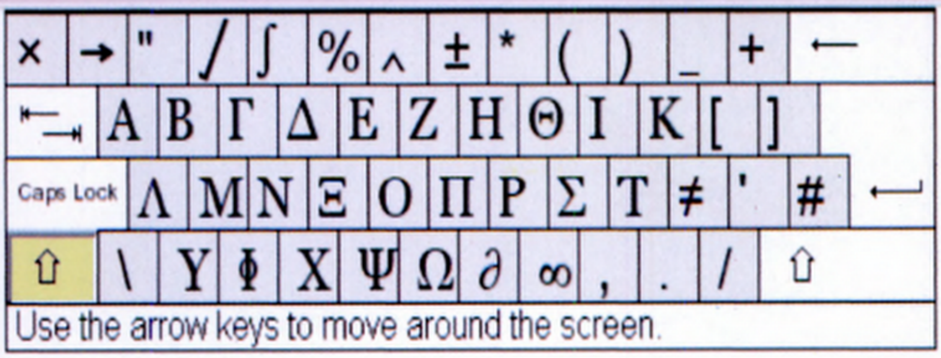

$$y_2 = \frac{\int_0^{\pi/2} \sin\theta \, d\theta}{\sqrt{x^2+1}}$$

Formulae and Definitions In Mathematics

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Table of Contents

Formulae and Definitions in Engineering Mathematics

1) Trigonometric Formulae

a) Trigonometric Formulae

1) $\sin^2 A + \cos^2 A = 1$

2) $\sec^2 A = 1 + \tan^2 A$

3) $\operatorname{cosec}^2 A = 1 + \cot^2 A$

4) $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

5) $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

6) $\tan(A \pm B) = \dfrac{\tan A \pm \tan B}{1 \mp \tan A \cdot \tan B}$

7) $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

8) $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

9) $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

10) $\cos C - \cos D = -2 \sin \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

11) $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

12) $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$

13) $2 \cos A \cos B = \cos(A-B) + \cos(A+B)$

14) $\sin 2A = 2 \sin A \cos A = \dfrac{2 \tan A}{1 + \tan^2 A}$

15) $\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A = \dfrac{1 - \tan^2 A}{1 + \tan^2 A}$

16) $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

17) $\sin 3A = 3 \sin A - 4 \sin^3 A$

18) $\cos 3A = 4 \cos^3 A - 3 \cos A$

19) $\tan 3A = \dfrac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$

20) $$\sin^2 x = \frac{1-\cos 2x}{2}$$

21) $$\cos^2 x = \frac{1+\cos 2x}{2}$$

22) $\sin^{-1}x+\cos^{-1}x = \pi/2$

23) $\tan^{-1}x+\cot^{-1}x = \pi/2$

24) $\text{cosec}^{-1}x+\sec^{-1}x = \pi/2$

b) Allied Angles Formulae

Angle	$-\theta$	$\pi/2-\theta$	$\pi/2+\theta$	$\pi-\theta$	$\pi+\theta$	$2\pi-\theta$	$2n\pi+\theta$
sin	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\sin\theta$	$\sin\theta$
cos	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$\cos\theta$	$\cos\theta$
tan	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$-\tan\theta$	$\tan\theta$

c) Sine Formulae

1) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$$

d) Cosine Formulae

1) $$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

2) $$\cos B = \frac{c^2+a^2-b^2}{2ca}$$

3) $$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

e) Projection Formulae

If a, b, c are sides of triangle and A, B, C are angles

1) $a = b\cos C + c\cos B$

2) $b = c\cos A + a\cos C$

3) $c = a\cos B + b\cos A$

2) Logarithm

1) If $a^x = N$ then $\log_a N = x$

2) $\log_a a = 1 \quad \log 1 = 0 \quad \log_e = 1$

3) $\log(mn) = \log m + \log n$

4) $\log\left(\dfrac{m}{n}\right) = \log m - \log n$

5) i) $\log_a b = \dfrac{1}{\log_b a}$

ii) $\log_b a = \dfrac{\log_e a}{\log_e b}$

6) $\log e^{(\text{any number})} = \text{any number}$

3) Differential Calculus

a) Limits

1) $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

2) $\lim\limits_{\theta \to 0} \dfrac{\tan\theta}{\theta} = 1$

3) $\lim\limits_{x \to 0} \dfrac{x^n - a^n}{x - a} = n\,a^{n-1}$

4) $\lim\limits_{x \to 0} \dfrac{a^x - 1}{x} = \log a$

5) $\lim\limits_{x \to 0} \dfrac{e^x - 1}{x} = 1$

6) $\lim\limits_{x \to 0} (1+x)^{1/x} = e$

7) $\lim\limits_{x \to \infty} \dfrac{k}{x^p} = 0 \quad (p > 0)$

b) Derivatives

1) $\dfrac{d}{dx}\sin x = \cos x$

2) $\dfrac{d}{dx}\cos x = -\sin x$

3) $\dfrac{d}{dx}\tan x = \sec^2 x$

4) $\dfrac{d}{dx}\cot x = -\operatorname{cosec}^2 x$

5) $\dfrac{d}{dx}\sec x = \sec x \tan x$

6) $\dfrac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$

7) $\dfrac{d}{dx}\log x = \dfrac{1}{x}$

8) $\dfrac{d}{dx}x^n = nx^{n-1}$

9) $\dfrac{d}{dx}e^x = e^x$

10) $\dfrac{d}{dx}a^x = a^x \log a$

11) $\dfrac{d}{dx}[f(x)]^n = n\,[f(x)]^{n-1} \cdot f'(x)$

12) $\dfrac{d}{dx}uv = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$

13) $\dfrac{d}{dx}\dfrac{u}{v} = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

14) $\dfrac{d}{dx}(uvw) = uv\dfrac{dw}{dx} + uw\dfrac{dv}{dx} + vw\dfrac{du}{dx}$

15) $\dfrac{d}{dx}u = \dfrac{du}{ds}\cdot\dfrac{ds}{dx}$ (Chain Rule)

c) Derivatives of Inverse Functions

1) $\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$

2) $\frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$

3) $\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

4) $\frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$

5) $\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$

6) $\frac{d}{dx}\text{cosec}^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$

4) Integration Formulae

a) Integration Formulae

1) $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c\ (n \neq -1)$

2) $\int \frac{1}{x}\,dx = \log x + c$

3) $\int e^x\,dx = e^x$

4) $\int a^x\,dx = \frac{a^x}{\log a}$

5) $\int \log x\,dx = x\log x - x$

6) $\int \sin x\,dx = -\cos x$

7) $\int \cos x\,dx = \sin x$

8) $\int \tan x\,dx = \log \sec x$

9) $\int \cot x\,dx = \log \sin x$

10) $\int \sec^2 x\,dx = \tan x$

11) $\int \text{cosec}^2 x\,dx = -\cot x$

12) $\int \sec x \tan x\,dx = \sec x$

13) $\int \text{cosec}\,x \cot x\,dx = -\text{cosec}\,x$

14) $\int \sec x\,dx = \log(\sec x + \tan x)$

15) $\int \text{cosec}\,x\,dx = \log(\text{cosec}\,x - \cot x)$

16) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$

17) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left(x+\sqrt{x^2-a^2}\right)$

18) $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left(x+\sqrt{x^2+a^2}\right)$

19) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right|$

20) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|$

21) $\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$

22) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$

23) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left[x+\sqrt{x^2-a^2}\right]$

24) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left[x+\sqrt{x^2+a^2}\right]$

25) $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

26) $\int \frac{f'(x)}{f(x)}\, dx = \log f(x)$

27) $\int \frac{f'(x)}{\sqrt{f(x)}}\, dx = 2\sqrt{f(x)}$

28) $\int [f(x)]^n f'(x)\, dx = \frac{[f(x)]^{n+1}}{n+1}$

29) $\int e^{f(x)} f'(x)\, dx = e^{f(x)}$

30) $\int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

31) $\int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

32) $\int u \cdot v \cdot dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx\right] dx$

33) $\int uv\, dx = u v_1 - u' v_2 + u'' v_3 \ldots$

b) Definite Integrals

1) $\int_a^b f(x)\, dx = \int_a^b f(t)\, dt$

2) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

3) $\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$

4) $\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx,\ a < c < b$

5) $\int_0^{2a} f(x)\, dx = \int_0^a [f(x) + f(2a-x)]\, dx$

6) $\int_{-a}^{a} f(x)\, dx = \begin{cases} 2\int_0^a f(x)\, dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$

5) Pascal's Triangle

Pascal's triangle gives the coefficients of the terms in the expansion of $(a+b)^n$

$(a+b)^1$: 1 1

$(a+b)^2$: 1 2 1

$(a+b)^3$: 1 3 3 1

$(a+b)^4$: 1 4 6 4 1

6) a) Complex Numbers

a)

A number of the form z = x + iy, where x and y are real numbers and $i=\sqrt{-1}$ is called complex number.

If z = x + iy is a complex number then x is called real part of z and is denoted by Re (z) and y is called imaginary part of z and is denoted by Im (z).

If Re (z) = 0 then z = iy is called purely imaginary number.

If Im (z) = 0 then z = x is called purely real number.

b) Demoivre's Theorem

$$(\cos\theta+i\sin\theta)^n = \cos n\theta+i\sin n\theta$$

c) Euler's Formulae

1) $e^{ix} = \cos x + i\sin x$

2) $e^{-ix} = \cos x - i\sin x$

3) $\cos x = \dfrac{e^{ix}+e^{-ix}}{2}$

4) $\sin x = \dfrac{e^{ix}-e^{-ix}}{2i}$

d) Logarithms of Complex Numbers

1) $\log(x+iy) = \log\sqrt{x^2+y^2} + i\left(2n\pi+\tan^{-1}\dfrac{y}{x}\right)$

2) $\log(x+iy) = \log\sqrt{x^2+y^2} + i\tan^{-1}\dfrac{y}{x}$

e) Hyperbolic Functions

1) $\cosh x = \dfrac{e^x+e^{-x}}{2}$

2) $\sinh x = \dfrac{e^x-e^{-x}}{2}$

3) $\sin(ix) = i\sinh x$

4) $\sinh(ix) = i\sin x$

5) $\cos(ix) = \cosh x$

6) $\cosh(ix) = \cos x$

7) $\tan(ix) = i\tanh x$

8) $\tanh(ix) = i\tan x$

9) $\cot(ix) = -i\coth x$

10) $\coth(ix) = -i\cot x$

11) $\sec(ix) = \text{sech}\, x$

12) $\text{sech}(ix) = \sec x$

13) $\text{cosec}(ix) = -i\,\text{cosech}\, x$

14) $\text{cosec}(ix) = -i\,\text{cosec}\, x$

f) Osborn's Rule

This rule is useful for finding hyperbolic formulae by using trigonometric formulae.

In any trigonometric formula, replace the trigonometric function by hyperbolic function and change the sign of the term involving product of two sines.

1) $\cosh^2 x - \sinh^2 x = 1$

2) $1 - \tanh^2 x = \operatorname{sech}^2 x$

3) $\cosh 2x = \cosh^2 x + \sinh^2 x$

4) $\sinh 2x = 2 \sinh x \cosh x$

5) $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$

6) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$

g) Inverse Hyperbolic Function

1) $\sinh^{-1} x = \log\left(x + \sqrt{x^2+1}\right)$

2) $\cosh^{-1} x = \log\left(x + \sqrt{x^2-1}\right)$

3) $\tanh^{-1} x = \frac{1}{2} \log\left|\frac{1+x}{1-x}\right|$

h) Derivatives of Hyperbolic Functions

1) $\frac{d}{dx}\sinh x = \cosh x$ 2) $\frac{d}{dx}\cosh x = \sinh x$

3) $\frac{d}{dx}\tanh x = \sec h^2 x$ 4) $\frac{d}{dx}\coth x = -\operatorname{cosech}^2 x$

5) $\frac{d}{dx}\operatorname{sech} x = -\sec hx \tanh x$ 6) $\frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x$

7) Successive Differentiation

a) Successive Differentiation

1) $\frac{d^n}{dx^n}[a^x] = a^x (\log_e a)^n$

2) $\frac{d^n}{dx^n}[e^{ax}] = a^n e^{ax}$

3) $y = (ax+b)^m$, where m is any real number

$$\frac{d^n}{dx^n}\left[(ax+b)^m\right] = m(m-1)\ldots(m-n+1)\, a^n (ax+b)^{m-n}$$

4) $y = (ax+b)^{-m}$, where m is positive

$$\frac{d^n}{dx^n}\left[\frac{1}{(ax+b)^m}\right] = \frac{(-1)^n (m+1)(m+2)\ldots(m+n-1)\, a^n}{(ax+b)^{m+n}}$$

Special Cases

Case I : If m is positive integer and m > n $\frac{d^n}{dx^n}\left[(ax+b)^m\right] = \frac{m!\, a^n (ax+b)^{m-n}}{(m-n)!}$

Case II : m = n $\frac{d^n}{dx^n}\left[(ax+b)^n\right] = n!\; a^n$

Case III : $\frac{d^n}{dx^n}\left[(ax+b)^m\right] = 0$ If m is positive integer and m < n

Case IV : $\frac{d^n}{dx^n}\left[\frac{1}{ax+b}\right] = \frac{(-1)^n\, n!\, a^n}{(ax+b)^{n+1}}$

1) $\frac{d^n}{dx^n}[\log(ax+b)] = \frac{(-1)^{n-1}(n-1)!\, a^n}{(ax+b)^n}$

2) $\frac{d^n}{dx^n}[\sin(bx+c)] = b^n \sin\left(bx+c+\frac{n\pi}{2}\right)$

3) $\frac{d^n}{dx^n}[\cos(bx+c)] = b^n \cos\left(bx+c+\frac{n\pi}{2}\right)$

4) $\frac{d^n}{dx^n}\left[e^{ax}\sin(bx+c)\right] = r^n\, e^{ax}\, \sin(bx+c+n\theta)$

5) $\frac{d^n}{dx^n}\left[e^{ax}\cos(bx+c)\right] = r^n\, e^{ax}\, \cos(bx+c+n\theta)$

where $r = \sqrt{a^2+b^2}$, $\theta = \tan^{-1}\frac{b}{a}$

b) Leibnitz's Theorem

$$(uv)_n = {}^nc_0\, u_n\, v + {}^nc_1\, u_{n-1}\, v_1 + {}^nc_2 u_{n-2}\, v_2 + \ldots + {}^nc_r\, u_{n-r}\, v_r + \ldots + {}^nc_n\, u\, v_n$$

8) Expansions of Functions

a) Taylor's Theorem

1) $f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!}\, f''(x) + \frac{h^3}{3!}\, f'''(x) + \ldots$

2) $f(x+h) = f(h) + x\, f'(h) + \frac{x^2}{2!}\, f''(h) + \frac{x^3}{3!}\, f'''(h) + \ldots$

3) $$f(x) = f(h) + (x-h)\, f'(h) + \frac{(x-h)^2}{2!}\, f''(h) + \ldots$$

b) Maclaurin's Theorem

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!}\, f''(0) + \ldots \frac{x^n}{n!}\, f^n(0) + \ldots$$

c) Indeterminate Forms

There are seven indeterminate forms viz. $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, $\infty - \infty$, 0^0, ∞^0, 1^∞

L'Hospital's rule : $\frac{0}{0}$, $\frac{\infty}{\infty}$

if $\lim\limits_{x \to a} f(x) = 0$ or ∞ and $\lim\limits_{x \to a} g(x) = 0$ or ∞

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

d) Standard Expansions

1) $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

2) $$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \ldots$$

3) $$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

4) $$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

5) $$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17}{315} x^7 + \ldots$$

6) $$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots$$

7) $$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots$$

8) $$\tanh x = x - \frac{x^3}{3} + \frac{2}{15} x^5 - \frac{17}{315} x^7 \ldots$$

9) $$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots$$

10) $$\log(1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \ldots$$

11) $(1+x)^n = 1+nx+\frac{n(n-1)}{2!}x^2+\ldots$

12) $\frac{1}{1+x} = 1-x+x^2-x^3+\ldots$

13) $\frac{1}{1-x} = 1+x+x^2+x^3+\ldots$

14) Binomial expansion

$$(x+a)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, a + {}^nC_2\, x^{n-2}\, a^2 + {}^nC_3\, x^{n-3}\, a^3 \ldots$$

$$\ldots\ldots + {}^nC_r\, x^{n-r}\, a^r + \ldots\ldots + {}^nC_n\, a^n$$

where $${}^nC_r = \frac{n!}{r!\,(n-r)!}$$

15) $1+2+3\ldots+n = \frac{n(n+1)}{2}$

16) $1^2+2^2+3^2\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$

17) $1^3+2^3+\ldots+n^3 = \left[\frac{n(n+1)}{2}\right]^2$

9) Sequences

A function whose domain is the set of natural numbers and range a set of real numbers is called a real sequence. i.e a order set of numbers $a_1, a_2, \ldots a_n$ is called a sequence and is denoted by $\{a_n\}$ or (a_n) if the number of terms are infinite it is called infinite sequence.

a) Bounds of a Sequence

a) Bounded above sequences

A sequence $\{a_n\}$ is said to be bounded above if there exists a real number k such that $S_n \leq K$ for all $n \in N$

b) Bounded below sequences

A sequence $\{a_n\}$ is said to be bounded below if there exists a real number k such that $S_n \geq k$ for all $n \in N$

c) Bounded sequences

A sequence is said to be bounded if it is both bounded below and bounded above.

b) Limits of a Sequence

Definition

A sequence $\{a_n\}$ is said have a limit '1' if for each $\in > 0$, there exists a positive integer N (depending on $\in$) such that $|a_n - 1| < \in$, for all $\geq N$ i.e. the term approach the value 1 as n becomes larger and larger.

i.e. $a_n \to 1$ as $n \to \infty$ or $\lim_{n \to \infty} a_n = 1$

c) Definition : Convergent Sequence

A sequence, which tends to finite limit is said to converge and is called convergent sequence.

Note

1) A limit of a sequence if it exists is unique.

2) Every convergent sequence is bounded.

3) Every bounded sequence need not be convergent. $(-1)^n$ is not convergent sequence but it is bounded.

d) Definition : Monotonic Sequence

1) A sequence $\{a_n\}$ is said to be monotonically increasing if $a_n \leq a_{n+1}$ for all n.

2) A sequence $\{a_n\}$ is said to be monotonically decreasing if $a_n \geq a_{n+1}$ for all n.

A sequence which is either monotonic increasing or monotonic decreasing is called monotonic sequence.

Note

1) A monotonic increasing sequence, and bounded above, is convergent.

2) A monotonic decreasing sequence, and bounded below, is convergent.

3) A monotonic increasing sequence, and not bounded above, diverges to ∞.

4) A monotonic decreasing sequence, and not bounded below, diverges to $-\infty$.

e) Limits of some Standard Sequences

1) $$\lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

2) $$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x \text{ for all } x$$

3) $$\lim_{n \to \infty} r^n = \begin{cases} 0 & \text{if } -1 < r < 1 \\ \infty & \text{if } r > 1 \end{cases}$$

Note

If $\{a_n\}$ and $\{b_n\}$ are two convergent sequences such that

$$\lim_{n \to \infty} a_n = A \text{ and } \lim_{n \to \infty} b_n = B$$

Then

1) $\{a_n + b_n\}$ is convergent and converges to A + B.

2) Sequences $\{a_n \cdot b_n\}$ is convergent and converges to A.B.

3) Sequence $\left\{\frac{a_n}{b_n}\right\}$ is convergent and converges to $\frac{A}{B}$ (provided $B \neq 0$).

10) Infinite Series

If $\{a_n\}$ be a sequence of real numbers, then $a_1 + a_2 + ... a_n$ is called an infinite series.

It is denoted by $\sum_{n=1}^{\infty} a_n$ or simply by $\sum a_n$.

A sequence $\{S_n\}$, where S_n, denote the sum of first n-terms of the series i.e. $S_n = a_1 + a_2 + a_3 + ... a_n$ for all n.

The sequence $\{S_n\}$ is called sequence of partial sums.

$S_1 = a_1$ $\quad S_2 = a_1 + a_2$ $\quad S_3 = a_1 + a_2 + a_3$

The series $\sum_{n=1}^{\infty} a_n$ is said to be convergent if the sequence of partial sums $\{S_n\}$ is convergent.

1) If $\lim_{n \to \infty} S_n$ = finite, then $\sum a_n$ is said to be convergent.

2) If $\lim_{n \to \infty} S_n \to \pm \infty$, then $\sum a_n$ is said to be divergent and diverges to $\pm \infty$.

3) If $\lim_{n \to \infty} S_n$ is not unique as $n \to \infty$, then $\sum a_n$ is said to be oscillatory or non-convergent.

Before discussing the details of convergence or divergence of an infinite series, the following properties should be noted.

1) Convergence and divergence of an infinite series remains unaffected by the addition or removal of finite number of terms from the series.

2) If a series in which all the terms are positive is convergent the series remains convergent when some or all of its terms are changed to negative.

3) If $\sum_{n=1}^{\infty} a_n$ is convergent or divergent series then $\sum_{n=1}^{\infty} k a_n$ will remain respectively convergent or divergent.

Note

Test for divergence

If $\lim_{n \to \infty} a_n \neq 0$, then $\sum a_n$ is divergent.

Tests for Convergence

a) Cauchy's Root Test

If $\sum a_n$ is a positive series such that $\lim_{n \to \infty} (a_n)^{\frac{1}{n}} = L$, then the series

i) Converges if $L < 1$

ii) Diverges if $L > 1$

iii) The test fails if $L = 1$.

b) Comparison Test

1) If two positive term series $\sum u_n$ and $\sum v_n$ such that

   i) $\sum v_n$ converges

   ii) $u_n \leq v_n$ for all values of n then $\sum u_n$ also converges.

2) If two positive term series $\sum u_n$ and $\sum v_n$ be such that

   i) $\sum v_n$ diverges

   ii) $u_n \geq v_n$ for all n then $\sum u_n$ also diverges

3) Limit form

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be such that $\lim_{n \to \infty} \frac{u_n}{v_n}$ = finite quantity ($\neq 0$).

Then $\sum u_n$ and $\sum v_n$ converge or diverge together.

i.e. If $\sum u_n$ is a series which is convergent then $\sum v_n$ is convergent and vice a versa.

Also if $\sum u_n$ is divergent then $\sum v_n$ is divergent and vice a versa.

If $\lim_{n \to \infty} \frac{u_n}{v_n} = 0$, test fails.

Working rule to find the auxiliary series $\sum_{n=1}^{\infty} v_n$

1) Find u_n and note that u_n contains the power of n only which may be positive or negative, integral or fractional.

2) If u_n is in the form of a fraction then we take $v_n = \frac{n^p}{n^q} = \frac{1}{n^{q-p}}$ where p and q are respectively the highest indices of n in the numerator and denominator of u_n.

3) If u_n can be expanded in ascending powers of $1/n$ then to get v_n, we should retain only the lowest power of $1/n$.

c) Ratio Test D'Alembert's Ratio Test

If $\sum_{n=1}^{\infty} u_n$ is a series of positive terms then it is convergent if $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} > 1$ and divergent if $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} < 1$.

If $\lim_{n \to \infty} \frac{u_n}{u_{n+1}} = 1$, the test fails and a further investigation will be required.

d) Raabe's Test

Let $\sum a_n$ be a series of positive terms and if $\lim_{n \to \infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = L$.

Then, i) $\sum a_n$ converges if $L > 1$

ii) $\sum a_n$ diverges if $L < 1$.

And test fails if $L = 1$.

Note

Raabe's test is stronger than ratio test. Normally this test is used if ratio test fails.

e) Cauchy's Condensation Test

If the function f(n) is positive for all positive integral values of n and continuously decreases as n increases, then the series $\sum_{n=1}^{\infty} f(n)$ is convergent or divergent according as $\sum_{n=1}^{\infty} a^n f(a^n)$ is convergent or divergent, "a" being a positive integer greater than unity.

Note : This test is applied when the series involves logarithmic expressions.

f) Gauss's Test

Let $\sum_{n=1}^{\infty} u_n$ be a series of positive terms and suppose that $\frac{u_n}{u_{n+1}}$ can be expressed in the form $\frac{u_n}{u_{n+1}} = 1 + \frac{l}{n} + \frac{b_n}{n^p}$, where $p > 1$ and b_n is bounded as $n \to \infty$ then $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$ and diverges if $l \leq 1$.

g) De Morgan's Bertrand's Test

The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent or divergent according as

$$\lim_{n \to \infty} \left[\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right] > 1 \text{ or } < 1.$$

Note : This test is to be applied when both D'Alembert's ratio test and Raabe's test fail.

Alternative form : The series $\sum_{n=1}^{\infty} u_n$ is convergent or divergent according as

$\lim_{n \to \infty} \left[\left(n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right] > 1$ or < 1.

h) Cauchy's Integral Test

Let $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} f(n)$ be a positive terms where f(n) decreases as n increases and let $I = \int_1^{\infty} f(x)\, dx$ then

i) If I is finite then $\sum_{n=1}^{\infty} u_n$ is convergent.

ii) If $I = \infty$ then $\sum_{n=1}^{\infty} u_n$ is divergent.

i) Logarithmic Test

The series $\sum_{n=1}^{\infty} u_n$ of positive terms is convergent or divergent according as $\lim_{n \to \infty} \left(n \log \frac{u_n}{u_{n+1}} \right) > 1$ or < 1.

Note : This test is an alternative to Raabe's test and is applied when D'Alembert's ratio test fails and when either

i) n occurs as an exponent in $\frac{u_n}{u_{n+1}}$ or

ii) Taking logarithm of $\frac{u_n}{u_{n+1}}$ makes the evaluation of limits easier.

j) Alternating Series

An infinite series in which the terms are alternately positive and negative is called an alternating series.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$ is an alternating series.

k) Leibnitz Test (Alternating Series Test)

An infinite series $\sum_{n=1}^{\infty} (-1)^{n-1} u_n$ in which the terms are alternately positive and negative is convergent if each term of the series is numerically less than the preceding term and $\lim_{n \to \infty} u_n = 0$.

l) Absolute Convergence

The series $\sum_{n=1}^{\infty} u_n$ which contains both the positive and negative terms is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |u_n|$ is convergent.

e.g. $\sum u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \ldots$ is an absolutely convergent series.

m) Conditional Convergence

If the alternating series $\sum_{n=1}^{\infty} u_n$ is convergent but the series $\sum_{n=1}^{\infty} |u_n|$ is divergent, then the series $\sum_{n=1}^{\infty} u_n$ is said to be conditionally convergent.

e.g. $\sum u_n = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ is convergent

and $\sum |u_n| = 1 + \frac{1}{2} + \ldots + \frac{1}{n} + \ldots$ is divergent.

Given series is conditionally convergent.

11) Matrices

a) Definition of Matrix

It is the arrangement of numbers in rows and columns.

It is denoted as $A(m \times n)$ i.e. A matrix with m rows and n columns.

Note - We can calculate the value of a determinant but not of a matrix.

b) Definition of Elementary Transformations

1) R_{ij}, C_{ij} : i.e. interchange of the elements of ith row (or column) with all the elements of jth row (or column).
2) KR_j, KC_j : i.e. multiplication by a scalar to every element of any row or column.
3) $R_i + KR_j$, $C_i + KC_j$: i.e. addition to the elements of ith row (or column) by the equimultiples of the elements of any other row (or column).

c) Definition of Rank of a Matrix

A number r greater than zero (r > 0) is said to be the rank of matrix A if

i) there exists at least one non-zero minor of order r,

ii) Every minor of order r + 1 is zero.

d) Minor of a Matrix

It is the determinant of a submatrix of a matrix.

e) Sub Matrix

It is the matrix obtained by deleting any rows and/or columns from the original matrix.

f) Definition of Normal Form

By performing elementary transformations on a matrix we can reduce it to one of the following forms known as the normal form.

$$[I_r],\ [I_r,\ 0],\ \begin{bmatrix} I_r \\ 0 \end{bmatrix},\ \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

where, 'I_r' is the unit matrix of order r, 'r' is the rank of matrix.

g) Definition of Inverse

If A is a nonsingular square matrix then their exists a non singular matrix B such that $AB = BA = I$ then $B = A^{-1}$ B is called as inverse of A.

12) Fourier Series

Definition : If f(x) is a periodic function with period 2π, defined in the interval $c \le x \le c + 2\pi$ and satisfies the Dirichlet's conditions then f(x) can be represented by a series

$$f(x) = \frac{a_0}{2} + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots$$

i.e.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

This representation of f(x) is called Fourier series and a_0, a_n, b_n are called the Fourier coefficients.

Note 1 : $\sin^{-1} x$ cannot have a Fourier series expansion in any interval, since it is not a single valued function.

Note 2 : tan x cannot have a Fourier series expansion in $(0, 2\pi)$, since it becomes infinite at $x = \pi / 2$.

Dirichlet's conditions

i) f(x) and its integrals are finite and ˡe valued.

ii) f(x) has at most finite number of fin..e discontinuities.

iii) f(x) has at most finite number of maxima and minima.

Definition : Let f(x) be defined in (– c, c).

1) If f (– x) = f(x), for all x in (– c, c) then f(x) is **even** function.

2) If f (– x) = – f(x), for all x in (– c, c) then f(x) is an **odd** function.

Note 1 : If f(x) is even, then its graph is symmetrical about Y axis.

Note 2 : If f(x) is odd, then its graph is symmetrical about opposite quadrants.

Note 3 : $\int_{-c}^{c} f(x)\,dx = 2\int_{0}^{c} f(x)\,dx$ if f(x) is an even function of x.

$= 0$ if f(x) is an odd function of x.

Note 4 : sin x and tan x are always odd functions, cos x is always even function.

Note 5 : check the function f(x) for evenness or oddness only if it is defined in $(-\pi, \pi)$ or $(-l, l)$.

Note : If n is any positive integer then,

1) $\cos n\pi = (-1)^n$, $\sin n\pi = 0$,

2) $\cos(2n \pm 1)\pi = -1$, $\sin(2n \pm 1)\pi = 0$

3) $\cos 2n\pi = 1$, $\sin 2n\pi = 0$

4) $(-1)^n = (-1)^{-n}$ $\therefore \cos(n+1)\pi = \cos(n-1)\pi$

5) $\cos\frac{n\pi}{2}$, $\sin\frac{n\pi}{2}$ depends on n.

Name of the series and interval	Fourier constants	Expression for the Fourier series
Fourier series $0 \le x \le 2\pi$	$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
Fourier series $-\pi \le x \le \pi$	$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Fourier series $-\pi \le x \le \pi$ even function Half range cosine series $0 \le x \le \pi$	$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$ $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$ $b_n = 0$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx)$
Fourier series $-\pi \le x \le \pi$ odd function Half range sine series $0 \le x \le \pi$	$a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$	$f(x) = \sum_{n=1}^{\infty}(b_n \sin nx)$
Fourier series general formula $c \le x \le c + 2\pi$	$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx$ $a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx$ $b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$
Fourier series $0 \le x \le 2L$	$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx$ $a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$
Fourier series $-L \le x \le L$	$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx$ $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$
Fourier series $-L \le x \le L$ even function Half range cosine series $0 \le x \le L$	$a_0 = \frac{2}{L}\int_0^{L} f(x)\,dx$ $a_n = \frac{2}{L}\int_0^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $b_n = 0$	$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right)\right)$

Fourier series $-L \le x \le L$ odd function Half range sine series $0 \le x \le L$	$a_0 = 0$ $a_n = 0$ $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$	$f(x) = \sum_{n=1}^{\infty}\left(b_n \sin\left(\frac{n\pi x}{L}\right)\right)$

a) Important Formulae for Fourier Series

1) $\cos n\pi = (-1)^n$

2) $\sin n\pi = 0$

3) $\int x \sin nx\,dx = \left[(x)\left(\frac{-\cos nx}{n}\right)-(1)\left(\frac{-\sin nx}{n^2}\right)\right]$

4) $\int x \cos nx\,dx = \left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(\frac{-\cos nx}{n^2}\right)\right]$

5) $\int x^2 \sin nx\,dx = \left[x^2\left(\frac{-\cos nx}{n}\right)-(2x)\left(\frac{-\sin nx}{n^2}\right)+2\left(\frac{\cos nx}{n^3}\right)\right]$

6) $\int x^2 \cos nx\,dx = \left[x^2\left(\frac{\sin nx}{n}\right)-(2x)\left(\frac{-\cos nx}{n^2}\right)+2\left(\frac{-\sin nx}{n^3}\right)\right]$

7) $\int e^{ax} \sin nx\,dx = \frac{e^{ax}}{a^2+n^2}[a \sin nx - n \cos nx]$

8) $\int e^{ax} \cos nx\,dx = \frac{e^{ax}}{a^2+n^2}[a \cos nx + n \sin nx]$

9) $\int \sin nx\,dx = \frac{-\cos nx}{n}$

10) $\int \cos nx\,dx = \frac{\sin nx}{n}$

11) $\int \sin x \sin nx\,dx = \frac{1}{2}\int \cos(1-n)x - \cos(1+n)x\,dx$

12) $\int \cos x \cos nx\,dx = \frac{1}{2}\int \cos(1-n)x + \cos(1+n)x\,dx$

13) $\int \sin x \cos nx\,dx = \frac{1}{2}\int \sin(1+n)x + \sin(1-n)x\,dx$

14) $\int \cos x \sin nx\,dx = \frac{1}{2}\int \sin(n+1)x + \sin(n-1)x\,dx$

b) Important Integrals for Fourier Series

Integral	$\int_{0}^{2\pi}$	$\int_{0}^{\pi}$	$\int_{-\pi}^{\pi}$
$\int x \sin nx\, dx$	$\frac{-2\pi}{n}$	$\frac{-(-1)^n \pi}{n}$	$\frac{-2\pi(-1)^n}{n}$
$\int x \cos nx\, dx$	0	$\frac{(-1)^n - 1}{n^2}$	0
$\int x^2 \sin nx\, dx$	$\frac{-4\pi^2}{n}$	$\frac{-\pi^2(-1)^2}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]$	0
$\int x^2 \cos nx\, dx$	$\frac{4\pi}{n^2}$	$\frac{2\pi(-1)^n}{n^2}$	$\frac{4\pi(-1)^n}{n^2}$
$\int e^{ax} \sin nx\, dx$	$\frac{n}{a^2+n^2}\left[1-e^{2a\pi}\right]$	$\frac{n}{a^2+n^2}\left[1-e^{a\pi}(-1)^n\right]$	$\frac{-n(-1)^n\left[e^{a\pi}-e^{-a\pi}\right]}{(a^2+n^2)}$
$\int e^{ax} \cos nx\, dx$	$\frac{a}{a^2+n^2}\left[e^{2a\pi}-1\right]$	$\frac{a}{a^2+n^2}\left[e^{a\pi}(-1)^n - 1\right]$	$\frac{a(-1)^n\left[e^{a\pi}-e^{-a\pi}\right]}{(a^2+n^2)}$
$\int \sin nx\, dx$	0	$\frac{1-(-1)^n}{n}$	0
$\int \cos nx\, dx$	0	0	0
$\int \sin x \sin nx\, dx$ $n \neq 1$	0	0	0 If $n \neq 1$
$\int \cos x \cos nx\, dx$ $n \neq 1$	0	0	0
$\int \sin x \cos nx\, dx$ $n \neq 1$	0	$\frac{1+(-1)^n}{1-n^2}$	0
$\int \cos x \sin nx\, dx$ $n \neq 1$	0	$n\left[\frac{1+(-1)^n}{n^2-1}\right]$	0

13) Error Functions

1) $$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du$$

2) $$\operatorname{erf}_c(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$$

3) $$\operatorname{erf}(0) = 0, \quad \operatorname{erf}(\infty) = 1$$

4) $$\operatorname{erf}(x) + \operatorname{erf}_c(x) = 1$$

14) Beta and Gamma Functions

1) $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \, (n > 0)$$

2) $$\Gamma(n+1) = n\,\Gamma(n) = n! \quad \text{(if n integer)}$$

3) $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

4) $$\Gamma(p)\,\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad \text{if } 0 < p < 1$$

5) $$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx \quad m > 0 \quad n > 0$$

6) $$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2}\,\beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

7) $$\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$$

8) $$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

15) D.U.I.S.

1) If a and b are constants.

$$\frac{d}{d\alpha} \int_a^b f(x,\alpha)\, dx = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)\, dx$$

2) If $a = f_1(\alpha)$, $b = f_2(\alpha)$

$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx + \frac{db}{d\alpha}\times f(b,\alpha) - \frac{da}{d\alpha}\times f(a,\alpha)$$

16) Reduction Formulae

1) $$\int_0^{\pi/2} \cos^n x\,dx = \int_0^{\pi/2} \sin^n x\,dx = \frac{[(n-1)\text{ subtract } 2 \ldots 2 \text{ or } 1]}{[(n)\text{ subtract } 2 \ldots 2 \text{ or } 1]} \times \left(\frac{\pi}{2} \text{ if n is even}\right)$$

2) $$\int_0^{\pi/2} \sin^m \cos^n x\,dx = \frac{[(m-1)\ldots 2 \text{ or } 1][(n-1)\ldots 2 \text{ or } 1]}{[(m+n)\text{ subtract } 2 \ldots 2 \text{ or } 1]} \times \left(\frac{\pi}{2} \text{ if m, n both even}\right)$$

3) $$\int_0^{\pi/2} \sin^m x \cos x\,dx = \frac{1}{m+1}$$

4) $$\int_0^{\pi/2} \cos^m x \sin x\,dx = \frac{1}{m+1}$$

17) Conversion Formulae

1) $$\int_0^{2\pi} \sin^m x \cos^n x\,dx = \begin{cases} = 4\int_0^{\pi/2} \sin^m x \cos^n x\,dx & \text{if m, n both even} \\ = 0 & \text{otherwise} \end{cases}$$

2) $$\int_0^{\pi} \sin^m x \cos^n x\,dx = \begin{cases} = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx & \text{if n even for any m} \\ = 0 & \text{if n odd for any m} \end{cases}$$

18) Rectification

1) $y = f(x)$ $$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

2) $x = f(y)$ $$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

3) $x = \phi_1(t)$, $y = \phi_2(t)$ $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

4) $r = f(\theta)$ $$ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

5) $r = f(\theta)$ $\quad ds = \sqrt{1 + r^2\left(\frac{d\theta}{dr}\right)^2}\ dr$

19) Multiple Integration

a) Area / Mass / Volume

1) $A = \iint dx\,dy$ $\quad$ 2) $\text{Mass} = \iint \rho\ dx\,dy$

3) If $x = r\cos\theta$ $\quad y = r\sin\theta$ then $dx\,dy = r\,dr\,d\theta$

$$A = \iint r\,dr\,d\theta$$

4) $\text{Vol} = \iiint dx\,dy\,dz$

b) C.G. of Arc

$$\bar{x} = \frac{\int_{\text{int}} x\,dm}{\int_{\text{int}} dm} \qquad \bar{y} = \frac{\int_{\text{int}} y\,dm}{\int_{\text{int}} dm}$$

For arc $dm = \rho\,ds$

$$\therefore \quad \bar{x} = \frac{\int x\rho\,ds}{\int \rho\,ds} \qquad \bar{y} = \frac{\int y\rho\,ds}{\int \rho\,ds}$$

c) C.G. of Area (Lamina)

$$\bar{x} = \frac{\int_{\text{int}} x\,dm}{\int_{\text{int}} dm} \qquad \bar{y} = \frac{\int_{\text{int}} y\,dm}{\int_{\text{int}} dm}$$

For area $dm = \rho\,dx\,dy$

$$\therefore \quad \bar{x} = \frac{\iint x\rho\,dx\,dy}{\iint \rho\,dx\,dy} \qquad \bar{y} = \frac{\iint y\rho\,dx\,dy}{\iint \rho\,dx\,dy}$$

d) C.G. of Volume

$$\bar{x} = \frac{\int_{\text{int}} x\,dm}{\int_{\text{int}} dm} \qquad \bar{y} = \frac{\int_{\text{int}} y\,dm}{\int_{\text{int}} dm}$$

$$\bar{z} = \frac{\int_{int} z \, dm}{\int_{int} dm}$$

For volume $dm = \rho \, dx \, dy \, dz$

$$\therefore \quad \bar{x} = \frac{\iiint x\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}$$

$$\bar{y} = \frac{\iiint y\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}$$

$$\bar{z} = \frac{\iiint z\rho \, dx \, dy \, dz}{\iiint \rho \, dx \, dy \, dz}$$

e) Moment of Inertia

$$\text{M.I.} = \int_{int} p^2 \rho \, dm$$

Where p is the $\perp$ distance of (x, y) from the axis of M.I. and ρ is the density at (x, y).

1) M.I. of arc M.I. $= \int p^2 \rho \, ds$

2) M.I. of area (lamina) M.I. $= \iint p^2 \rho \, dx \, dy$

3) M.I. of volume M.I. $= \iint p^2 \rho \, dx \, dy \, dz$

f) Radius of Gyration

$$K = \sqrt{\frac{I}{M}}$$

Parallel axes theorem : If I_G is M.I. of a body about an axis through the centroid G of the body and I_A, the M.I. of the body about any parallel axis through any point A, then $\mathbf{I_A = I_G + M d^2}$ where M is mass of the body and d is the distance between the parallel axis.

Perpendicular axes theorem : In case of a plane lamina in the XOY plane $\mathbf{I_Z = I_X + I_Y}$ where I_X and I_Y are M.I. about OX and OY and I_Z, M.I. about z axis.

Note : This theorem is applicable to plane lamina and not to three dimensional solids.

g) Spherical Polar Co-ordinates

i) For sphere $x^2 + y^2 + z^2 = a^2$

$$x = r \sin\theta \cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$\therefore \quad dxdydz = r^2 \sin\theta \, dr\, d\phi\, d\theta$$

1) For sphere $x^2+y^2+z^2 = a^2$ $\quad \int_0^{\pi}\int_0^{2\pi}\int_0^{a} \ldots dr\, d\phi\, d\theta$

2) For hemisphere $\quad \int_0^{\pi/2}\int_0^{2\pi}\int_0^{a} \ldots dr\, d\phi\, d\theta$

3) For octant of sphere $\quad \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{a} \ldots dr\, d\phi\, d\theta$

ii) For Ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

$$x = a\,r\sin\theta\cos\phi$$

$$y = b\,r\sin\theta\sin\phi$$

$$z = c\,r\cos\theta$$

$$\therefore \quad dxdydz = abc \quad r^2\sin\theta\, dr\, d\phi\, d\theta$$

1) For full ellipsoid $\quad \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \ldots dr\, d\phi\, d\theta$

2) For hemi ellipsoid $\quad \int_0^{\pi/2}\int_0^{2\pi}\int_0^{1} \ldots dr\, d\phi\, d\theta$

3) For octant $\quad \int_0^{\pi/2}\int_0^{\pi/2}\int_0^{1} \ldots dr\, d\phi\, d\theta$

h) Mean Values and R.M.S. Values

1) $$\text{M.V.} = \frac{\int_a^b f(x)\,dx}{\int_a^b dx}$$

2) $$\text{M.V.} = \frac{\iint_R f(x\,y)\,dx\,dy}{\iint dx\,dy}$$

3) R.M.S. $= \sqrt{\dfrac{\int_a^b [f(x)]^2 \, dx}{\int_a^b dx}}$

4) R.M.S. $= \sqrt{\dfrac{\iint_R [f(x\,y)]^2 \, dx\,dy}{\iint_R dx\,dy}}$

20) Co-ordinate System, Plane, Straight Line

a) Distance Formula

$$P_1 Q_1 = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

b) Section Formula

1) Internal division

$$x = \frac{mx_2+nx_1}{m+n} \qquad y = \frac{my_2+ny_1}{m+n} \qquad z = \frac{mz_2+nz_1}{m+n}$$

$$x = \frac{\lambda x_2+x_1}{\lambda+1} \qquad y = \frac{\lambda y_2+y_1}{\lambda+1} \qquad z = \frac{\lambda z_2+z_1}{\lambda+1}$$

2) External division

$$x = \frac{mx_2-nx_1}{m-n} \qquad y = \frac{my_2-ny_1}{m-n} \qquad z = \frac{mz_2-nz_1}{m-n}$$

$$x = \frac{\lambda x_2-x_1}{\lambda-1} \qquad y = \frac{\lambda y_2-y_1}{\lambda-1} \qquad z = \frac{\lambda z_2-z_1}{\lambda-1}$$

c) Direction Cosines

Let the line OP be inclined at angles α, β, γ with XYZ axis respectively. Then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called direction cosines (d.c.'s) of the line OP and are denoted by l, m, n respectively. Then,

$$l^2+m^2+n^2 = 1$$

i) Direction Ratios

Any three numbers a, b, c that are proportional to the direction cosines l, m, n respectively are called direction ratios.

Hence if a, b, c be the d.r.'s. of a given line, then the actual d.c.'s. are

$$l = \frac{a}{\sqrt{a^2+b^2+c^2}} \qquad m = \frac{b}{\sqrt{a^2+b^2+c^2}} \qquad n = \frac{c}{\sqrt{a^2+b^2+c^2}}$$

d) Angle between Two Lines

If (l_1, m_1, n_1) and (l_2, m_2, n_2) are the direction cosines of the two lines. Then

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

If direction ratios of two lines are given as a_1, b_1, c_1 and a_2, b_2, c_2. Then

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

Conditions for perpendicularity and parallelism

1) When given lines are perpendicular $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ or

2) When given lines are parallel $\quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = 1 \qquad \text{or} \qquad \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

e) Projection of a Segment of a Line

$$P'Q' = PQ\cos\theta$$

$$P'Q' = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

21) The Plane

1) **General form :** $ax + by + cz + d = 0$

2) Equation of plane passing through (x_1, y_1, z_1) with a, b, c as dr's of the normal to the plane is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

3) **Intercept form :** $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

4) **Normal (perpendicular) form :** $lx + my + nz = p$

5) Three points form : The equation of the plane passing through three points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

a) Angle between Two Planes

$$a_1 x + b_1 y + c_1 z + d_1 = 0$$

$$a_2 x + b_2 y + c_2 z + d_2 = 0$$

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\ \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

If the planes are perpendicular then,

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

If the planes are parallel then,

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

b) Length of the Perpendicular

The length of the perpendicular from the point $P(x_1, y_1, z_1)$ on the plane $ax + by + cz + d = 0$ is,

$$p = \left| \frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}} \right|$$

22) The Line

a) Symmetrical Form

If the line passes through a fixed point (x_1, y_1, z_1) and has d.r.' s (a, b, c) its equation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

b) Two Point Formula

The straight line which passes through two fixed points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

c) Coplanarity of Lines

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1} \text{ and } \frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$$

Are coplanar if

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Differentiating the above equation partially with respect to x, y, z, t and then put t = 1, we get,

$$\frac{\partial F}{\partial x} = 0, \frac{\partial F}{\partial y} = 0, \frac{\partial F}{\partial z} = 0 \text{ and } \frac{\partial F}{\partial t} = 0$$

Solving these equations we get co-ordinates of vertex.

c) Right Circular Cone

A right circular cone is a surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed straight line through the vertex.

The fixed point is called **vertex**, the fixed line the **Axis** of the cone and the angle is known as the **semi-vertical angle** of the cone.

Note : The section of right circular cone by any plane perpendicular to its axis is circle.

d) Enveloping Cone

The locus of the tangent lines from a given point to a given surface is a cone and is called the enveloping cone of the surface with the given point at its vertex.

i) Equation of the Enveloping Cone

The equation of the enveloping cone of the sphere $x^2+y^2+z^2=a^2$ with vertex at the point (α, β, γ)

$$\begin{aligned} S &\equiv x^2+y^2+z^2-a^2 \\ S_1 &\equiv \alpha^2+\beta^2+\gamma^2-a^2 \\ T &\equiv \alpha x + \beta y + \gamma z - a^2 \text{ or} \\ SS_1 &= T^2 \end{aligned}$$

Note : If the given sphere is $S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$, then enveloping cone will still be found to be

$$\begin{aligned} SS_1 &= T^2 \\ S_1 &\equiv \alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d \\ T &\equiv x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d \end{aligned}$$

25) a) Cylinder

a)

A cylinder is a surface generated by a straight line which is always parallel to a fixed straight line and satisfies one more condition e.g. it intersects a given curve or touches a given surface.

c) Type II : Reducible to V.S. Form

1) $\dfrac{dy}{dx} = f(ax+by+c)$ put $ax+by+c=v$

2) $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$ put $\dfrac{y}{x} = v$

d) Type III : Homogeneous D.E.

General Form : $\dfrac{dy}{dx} = \dfrac{f_1(x,y)}{f_2(x,y)}$

The differential equation of the above form is said to be homogeneous if $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions in x and y of the same degree.

These equations are reducible to V.S. form by the substitution

$y = vx \quad \therefore \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

e) Type IV : Non-homogeneous D.E.

General form $\quad \dfrac{dy}{dx} = \dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ ... (1)

Case (i)

If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$ put the common factor = v

Case (ii)

If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$

In this case to reduce equation (1) to homogeneous form we substitute $x = X + h$, $y = Y + h$ where h and k are constants to be determined.

Also $dx = dX$, $dy = dY$

$\therefore \dfrac{dy}{dx} = \dfrac{dY}{dX}$ Equation (1) becomes,

$$\frac{dY}{dX} = \frac{a_1X+b_1Y+(a_1h+b_1k+c_1)}{a_2X+b_2Y+(a_2h+b_2k+c_2)}$$

Choose h and k such that equation will become homogeneous in X and Y i.e.

$a_1h+b_1k+c_1 = 0$ and $a_2h+b_2k+c_2 = 0$

We get $\dfrac{dY}{dX} = \dfrac{a_1X+b_1Y}{a_2X+b_2Y}$

Which is a homogenous equation in X and Y

f) Type V : Exact Differential Equations

General form : $M(x,y)\,dx + N(x,y)\,dy = 0$

If there exists a function u (x, y) such that M dx + N dy = du then the differential equation is called as an exact differential equation.

Condition of exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ its general solution is

$$\int_{y = \text{constant}} M\,dx + \int [\text{terms of N not containing x}]\,dy = c$$

OR

$$\int_{x = \text{constant}} N\,dy + \int [\text{terms of M not containing y}]\,dx = c$$

g) Type VI : Equations Reducible to Exact Integrating Factor

An equation, which is not exact, can be made exact by multiplying it by a suitable factor called as the integrating factor (I.F.).

Rule 1 : If the given differential equation is homogeneous, then

$$\text{I.F.} = \frac{1}{Mx + Ny}$$

Rule 2 : If the given D.E. has the form, $y \cdot f_1(xy)\,dx + xf_2(xy)dy = 0$ then,

$$\text{I.F.} = \frac{1}{Mx - Ny}$$

Rule 3 : If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$ (say) Then I.F. = $e^{\int f(x)\,dx}$

Rule 4 : If $\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = \phi(y)$ (say) Then, I.F. = $e^{\int \phi(y)\,dy}$

Rule 5 : If the equation M dx + N dy = 0 then I.F. can be written as I.F. = $x^{\alpha}\ y^{\beta}$

h) Type VII : Linear Differential Equation

A differential equation is said to be linear if the differential equation is of first degree.

General form $\frac{dy}{dx} + Py = Q$ P, Q are functions of x then general solution is $y\ e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx} \cdot dx + C$

General form $\frac{dx}{dy} + P \cdot x = Q$ P, Q are functions of y then general solution is $x \cdot e^{\int P dy} = \int Q e^{\int P dy} dy + C$

Type 2

X = sin ax or cos ax

1) $\dfrac{1}{f(D^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b) = \dfrac{1}{f(-a^2)} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b)$

2) $\dfrac{1}{(D^2+a^2)^r} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}(ax+b) = \left(\dfrac{-x}{2a}\right)^r \dfrac{1}{r!} \begin{smallmatrix}\sin\\ \cos\end{smallmatrix}\left(ax+b+\dfrac{r\pi}{2}\right)$

Note :

1) In this type replace D^2 by $-a^2$, D^3 by $-a^2D$ and D^4 by a^4. Don't replace D by $\sqrt{-a^2}$.
2) If the factor D remains in the denominator then rationalize to get D^2 in the denominator.
3) Here don't take the factors of f (D) for finding P.I. Take the factors only if the denominator becomes zero after the replacement of D^2 by $-a^2$.
4) If there is a combination of zero and nonzero factor then operate the nonzero factor first.
5) If D is a factor of f (D) then operate $\frac{1}{D}$ first.

Type 3

If $X = x^n$ where n is a positive integer then use the binomial series

1) $(1+z)^n = 1 + nz + \dfrac{n(n-1)}{2!} z^2 \ldots$

2) $\dfrac{1}{1+z} = 1 - z + z^2 \ldots$

3) $\dfrac{1}{1-z} = 1 + z + z^2 \ldots$

Note :

1) Don't take the factors of f (D) for finding P.I.
2) Take the least power term outside with its sign, then f (D) will take the form $(1+z)$ or $(1-z)$ then use the binomial series.
3) If D is factor of f (D) then don't operate $\frac{1}{D}$ first as it will increase the power of x.

Type 4

$$\frac{1}{f(D)} e^{ax} v = e^{ax} \frac{1}{f(D+a)} v$$

c) Dirac Delta/Unit Impulse Function

Consider the function,

$$U(t-a) = \begin{cases} 0 & t < a \\ \dfrac{1}{\varepsilon} & a \le t \le a+\varepsilon \\ 0 & t > a+\varepsilon \end{cases} \qquad \delta(t-a) = \lim_{\varepsilon \to 0} U(t-a)$$

1) $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$

2) $L\,\delta(t-a) = \int_0^\infty e^{-st}\,\delta(t-a)\,dt = e^{-as}$

3) $L\,f(t)\,\delta(t-a) = e^{-as} f(a)$

d) Inverse Laplace Transforms

	$\phi(s)$	$L^{-1}\phi(s) = f(t)$
1)	$\frac{1}{s}$	1
2)	$\frac{1}{s-a}$	e^{at}
3)	$\frac{1}{s^{n+1}}$	$\frac{t^n}{\lceil n+1}\ \frac{t^n}{n!}$ (n = integer)
4)	$\frac{1}{s^2+a^2}$	$\frac{1}{a}\sin at$
5)	$\frac{1}{s^2-a^2}$	$\frac{1}{a}\sinh at$
6)	$\frac{s}{s^2+a^2}$	$\cos at$
7)	$\frac{s}{s^2-a^2}$	$\cosh at$
8)	1	$\delta(t)$
9)	$\phi(s-a)$	$e^{at} f(t) = e^{at} L^{-1}\phi(s)$
10)	$\phi'(s)$	$-t\,f(t)$
11)	$\int_s^\infty \phi(s)\,ds$	$\frac{f(t)}{t}$
12)	$\frac{1}{s}\phi(s)$	$\int_0^t f(t)\,dt$

29) Vectors

a)

Let $\bar{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

where $\hat{i}$, $\hat{j}$, $\hat{k}$ are unit vectors in direction of X, Y. Z respectively.

a_1, a_2, a_3 are the components of $\bar{a}$ in direction of X, Y, Z axis respectively.

$$\hat{a} = \frac{\bar{a}}{|\bar{a}|}$$

$$|\bar{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$$

b) Dot product of two vectors

$$\bar{a}\cdot\bar{b} = ab\cos\theta$$

where $a = |\bar{a}|$, $b = |\bar{b}|$ and $\theta \to$ angle between 2 vectors $\bar{a}$ and $\bar{b}$.

Since cos 0 = 1 and cos 90 = 0 we have

1) $\therefore \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

2) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

3) $\therefore \bar{a}\cdot\bar{b} = a_1b_1 + a_2b_2 + a_3b_3$

4) $\bar{a}\cdot\bar{b} = \bar{b}\cdot\bar{a}$ Commutative

5) Dot product of 2 vectors is scalar.

c) Cross Product of Two Vectors

$$\bar{A}\times\bar{B} = ab\sin\theta\,\hat{n}$$

Where $\hat{n} \to$ unit vector perpendicular to the plane of $\bar{a}$ and $\bar{b}$ and

$$\hat{n} = \frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}$$

1) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$

2) $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

3) $\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

4) $(\bar{a}\times\bar{b}) = -(\bar{b}\times\bar{a}) \leftarrow$ not commutative

5) Cross product of 2 vectors is a vector.

xi) Standard results

1) $\frac{d}{dt}(\bar{u} \pm \bar{v}) = \frac{d\bar{u}}{dt} + \frac{d\bar{v}}{dt}$

2) $\frac{d}{dt}(\bar{u}) = \frac{d\bar{u}}{ds}\frac{ds}{dt}$

3) $\frac{d}{dt}(\bar{u}\phi) = \bar{u}\frac{d\phi}{dt} + \phi\frac{d\bar{u}}{dt}$

4) $\frac{d}{dt}(\bar{u} \cdot \bar{v}) = \frac{d\bar{u}}{dt} \cdot \bar{v} + \bar{u} \cdot \frac{d\bar{v}}{dt}$

5) $\frac{d}{dt}(\bar{u} \times \bar{v}) = \frac{d\bar{u}}{dt} \times \bar{v} + \bar{u} \times \frac{d\bar{v}}{dt}$

6) $\frac{d}{dt}[\bar{u} \quad \bar{v} \quad \bar{w}] = \left[\frac{d\bar{u}}{dt} \quad \bar{v} \quad \bar{w}\right] + \left[\bar{u} \quad \frac{d\bar{v}}{dt} \quad \bar{w}\right] + \left[\bar{u} \quad \bar{v} \quad \frac{d\bar{w}}{dt}\right]$

7) $\frac{d}{dt}\,\bar{u} \times (\bar{v} \times \bar{w}) = \frac{d\bar{u}}{dt} \times (\bar{v} \times \bar{w}) + \bar{u} \times \left(\frac{d\bar{v}}{dt} \times \bar{w}\right) + \bar{u} \times \left(\bar{v} \times \frac{d\bar{w}}{dt}\right)$

Component of a vector $\bar{a}$ in the direction of $\bar{u}$ is given by $\bar{a} \cdot \hat{u}$ i.e. $\frac{\bar{a} \cdot \bar{u}}{|\bar{u}|}$

h) Tangential and normal components of acceleration of a particle describing a plane curve in parametric co-ordinates.

Let a_T = Tangential component of acceleration and a_N = Normal component of acceleration.

$$a_T = \frac{\bar{a} \cdot \bar{v}}{|\bar{v}|}$$

$$\therefore \quad a_N = \frac{|\bar{a} \times \bar{v}|}{|\bar{v}|}$$

i) Radial and transverse components of velocity and acceleration of a particle describing a plane curve in polar co-ordinates.

Radial velocity = $\dot{r}$

Radial acceleration = $\ddot{r} - r\dot{\theta}^2$

Transverse velocity = $r\dot{\theta}$

Transverse acceleration = $r\ddot{\theta} + 2\dot{r}\dot{\theta}$

j) Law of Central Orbits (Orbital Motion)

Let a particle describes the curve $r = f(\theta)$ under the action of force which is always directed towards a fixed center o then, $f = h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right]$ gives the law of acceleration i.e. the law of force towards the pole where f represents the acceleration h is a constant and $u = 1/r$

m) Directional Derivatives

If $\phi(x, y, z)$ is a scalar point function then the component of grad in any direction is equal to the directional derivative or rate of change of ϕ in that direction

$$DD = (\nabla\phi)_p \cdot \hat{u}$$

where $\bar{u}$ is the given direction and p is the given point.

As component of vector is maximum in direction of vector itself therefore component of grad ϕ is maximum in the direction of grad ϕ hence grad ϕ represents maximum rate of change of ϕ i.e. DD is maximum in the direction of $\nabla\phi$ and maximum magnitude of $DD = |\nabla\phi|$

n) Divergence of a Vector Point Function

Let $\bar{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

$$\nabla \cdot \bar{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

which is a S.P.F. called the divergence of $\bar{F}$

If $\nabla \cdot \bar{F} = 0$ then the $\bar{F}$ is said to be solenoidal.

o) Curl of a Vector Point Function $\bar{F}$

$$\text{Curl } \bar{F} = \nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

If $\nabla \times \bar{F} = 0$ then $\bar{F}$ is said to be irrotational.

In this case there exists a scalar point function ϕ such that $\bar{F} = \nabla\phi \cdot \phi$ is known as scalar potential of $\bar{F}$ the formula for scalar potential is given by,

$$d\phi = F_1 dx + F_2 dy + F_3 dz$$

i.e.

$$\phi = \int_{y,\, z \text{ constant}} F_1 dx + \int_{z \text{ constant free } x} F_2 dy + \int_{\text{free } x,\, y} F_3 dz$$

Note :

1) Curl $\bar{F}$ is vector point function.
2) Div $\bar{F}$ is scalar point function.
3) When $\nabla \times \bar{F} = 0$ angular velocity = 0 hence irrotational.

q) Definitions

1) Laplacian of ϕ

The expression div $(\text{grad}\,\phi) = \nabla\cdot(\nabla\phi)$

$$= \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial\phi}{\partial x}+\hat{j}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2} \text{ is known as Laplacian of } \phi$$

Here the operator $\nabla^2 = \nabla\cdot\nabla = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is known as laplace's operator.

The equation $\nabla^2\phi = 0$ is known as Laplace's equation in 3-dimension

2) The group operator $\bar{a}\cdot\nabla$

$\bar{a}\cdot\nabla = a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}$ is the operator. Where $\bar{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

i) Results

i) $(\bar{a}\cdot\nabla)\,\bar{r} = \bar{a}$

ii) $(\bar{a}\cdot\nabla)\phi\,\bar{F} = \phi(\bar{a}\cdot\nabla)\,\bar{F}+\bar{F}(\bar{a}\cdot\nabla)\phi$

iii) $\nabla^2 f(r) = f''(r)+\frac{2}{r}f'(r)$

30) Vector Integration

a) Line Integral

The expression $\int_c \bar{F}\cdot d\bar{r}$ is known as line integral of $\bar{F}$ taken over curve C and it gives the work done in moving a particle along curve C.

If, $\bar{F} = F_1\hat{i}+F_2\hat{j}+F_3\hat{k}$ and $\bar{r} = x\hat{i}+y\hat{j}+z\hat{k}$ then $d\bar{r} = dx\,\hat{i}+dy\,\hat{j}+dz\,\hat{k}$ and $\bar{F}\cdot d\bar{r} = F_1dx+F_2dy+F_3dz$

b) Green's Theorem

If R is a plane region bounded by one or more closed curve, and if M and N are single valued continuous functions along the boundary C and at all the points in region R, then,

$$\int_c M\,dx+N\,dy = \iint_R\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dxdy$$

i) Vector Form of Green's Theorem

If $\overline{F} = M\hat{i} + N\hat{j}$ where M and N are the functions of x and y then,

$$\overline{F}\cdot d\overline{r} = M\,dx + N\,dy$$

$$\therefore \quad \nabla \times \overline{F} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$\therefore \quad \left(\nabla \times \overline{F}\right)\hat{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

$\therefore$ From equation (1)

$$\int_c \overline{F}\cdot d\overline{r} = \iint_R \left[\left(\nabla \times \overline{F}\right)\cdot k\right] dxdy$$

Note :

1) For double integral over the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

   Put $x = ar\cos\theta \quad y = br\sin\theta \quad dxdy = ab\,r\,dr\,d\theta$

   limits $\int_0^{2\pi}\int_0^1 ab\,r\,drd\theta$

2) For single integral over ellipse

   Put $x = a\cos\theta \quad y = b\sin\theta$ limits of $\theta \to 0$ to 2π

3) For double integral over the circle $x^2 + y^2 = a^2$

   Put $x = r\cos\theta \quad y = r\sin\theta \quad dxdy = rdrd\theta$

   limits $\int_0^{2\pi}\int_0^a r\,drd\theta$

4) For single integral over the circle,

   Put $x = a\cos\theta \quad y = a\sin\theta$ Limits $\theta \to 0$ to 2π

c) Stroke's Theorem

It is the generalization of vector form of Green's theorem.

Statement Stroke's Theorem

The surface integral of normal component of curl of $\overline{F}$ taken over the surface S is equal to the line integral of tangential component of $\overline{F}$ around perimeter of curve C bounding open surface S.

$$\int_c \overline{F}\cdot d\overline{r} = \iint_s \left(\nabla \times \overline{F}\right)\cdot \hat{N}\,ds$$

Note :

1) If the surface is closed then the bounding curve does not exist therefore we cannot evaluate Stroke's theorem over the closed surface.
2) If $\overline{F}$ is irrotational then

$$\nabla \times \overline{F} = 0$$

$$\therefore \quad \int_c \overline{F}\cdot d\overline{r} = 0$$

3) The abbreviation of $\hat{N}$ ds is $d\overline{s}$.
4) If S_1 and S_2 are 2 surfaces with the same bounding curve 'C' then

$$\iint_{s_1} \left(\nabla \times \overline{F}\right)\cdot\overline{ds_1} = \iint_{s_2} \left(\nabla \times \overline{F}\right)\cdot\overline{ds_2}$$

5) If the surface is in X – O – Y plane then, $\hat{N} = \hat{k}$, ds = dxdy
6) If surface is not X – O – Y plane then, $\hat{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$ then use projection formula

$$ds = \frac{dxdy}{\left|\hat{N}\cdot\hat{k}\right|}$$

projection of surface element ds on X – O – Y plane.

d) Gauss's Divergence Theorem

The surface integral of the normal component of a vector $\overline{F}$ taken over a closed surface and enclosing volume V is equal to the volume integral of divergence of $\overline{F}$ taken throughout volume V bounded by surface S.

$$\iint_{s} \overline{F}\cdot\hat{N}\, ds = \iiint_{v} \left(\nabla\cdot\overline{F}\right) dv$$

31) Fourier Transforms

Fourier transform of a non periodic function f (t) in time domain into a function F(λ) in frequency domain.

Definition

Let f (x) be defined in (– c, c).

3) If f (– x) = f (x), for all x in (– c, c) then f (x) is **even** function.
4) If f (– x) = – f (x), for all x in (– c, c) then f (x) is an **odd** function.

Note 1 : If f (x) is even, then its graph is symmetrical about Y-axis.

Note 2 : If f (x) is odd, then its graph is symmetrical about opposite quadrants

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